Addendum to "The Dirac-Coulomb Sturmians and the series expansion of the Dirac-Coulomb Green function: application to the relativistic polarizability of the hydrogen-like atom" [J. Phys. B: At. Mol. Opt. Phys. **30** (1997) 825–61, (E) **30** (1997) 2747]

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Abstract

Closure relations satisfied by the radial Dirac-Coulomb Sturmians are proved analytically. The Sturmian expansion of the Dirac-Coulomb Green function is transformed to the form containing only series with summations running over non-negative indices. The main paper was published in *J. Phys. B: At. Mol. Opt. Phys.* **30** (1997) 825–61 [Erratum: **30** (1997) 2747], see also *J. Phys. A: Math. Gen.* **31** (1998) 4963–90 [Erratum: **31** (1998) 7415–6].

1 Proofs of the closure relations

In this section it is our goal to prove correctness of the following closure relations satisfied by the Dirac-Coulomb Sturmians (equations (91) and (26) of Ref. [1])

$$\frac{\alpha^{-1}}{2}\sum_{n=-\infty}^{\infty}\begin{pmatrix} \varepsilon S_{n\kappa}(x) \\ \varepsilon^{-1}T_{n\kappa}(x)\end{pmatrix}\begin{pmatrix} S_{n\kappa}(x') & T_{n\kappa}(x')\end{pmatrix}=\delta(x-x')\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{1}$$

and

$$\frac{Z}{x}\sum_{n=-\infty}^{\infty}\begin{pmatrix} S_{n\kappa}(x) \\ -\mu_{n\kappa}^{-1}T_{n\kappa}(x)\end{pmatrix}\begin{pmatrix} \mu_{n\kappa}S_{n\kappa}(x') & T_{n\kappa}(x')\end{pmatrix}=\delta(x-x')\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{2}$$

In the proofs we shall use the following forms of the upper and lower components of the radial Sturmians (equations (60) and (61) of Ref. [1])

$$S_{n\kappa}(x)=\sqrt{\frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}}\,x^{\gamma_\kappa}\mathrm{e}^{-x/2}\left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right], \tag{3}$$

$$T_{n\kappa}(x)=\sqrt{\frac{\alpha\varepsilon(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}}\,x^{\gamma_\kappa}\mathrm{e}^{-x/2}\left[L_{|n|-1}^{(2\gamma_\kappa)}(x)-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right] \tag{4}$$

(the upper signs should be taken for $n>0$ and the lower signs for $n<0$; for $n=0$ one takes the upper sign if $\kappa<0$ and the lower one if $\kappa>0$), the expression for the eigenvalue $\mu_{n\kappa}$ associated with the Sturmian $\begin{pmatrix} S_{n\kappa}(x) & T_{n\kappa}(x)\end{pmatrix}^{\top}$ (equation (52) of Ref. [1])

$$\mu_{n\kappa}=\varepsilon\zeta(|n|+\gamma_\kappa\pm N_{n\kappa}) \tag{5}$$

and the well-known closure relation for the generalized Laguerre polynomials

$$x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+2\gamma_\kappa+1)}L_n^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa)}(x')=\delta(x-x'). \tag{6}$$

Consider equation (1). It is equivalent to four relations

$$\frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty}S_{n\kappa}(x)S_{n\kappa}(x')=\delta(x-x'), \tag{7}$$

$$\frac{\alpha^{-1}\varepsilon^{-1}}{2}\sum_{n=-\infty}^{\infty}T_{n\kappa}(x)T_{n\kappa}(x')=\delta(x-x'), \tag{8}$$

$$\frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty}S_{n\kappa}(x)T_{n\kappa}(x')=0 \tag{9}$$

and

$$\frac{\alpha^{-1}\varepsilon^{-1}}{2}\sum_{n=-\infty}^{\infty}T_{n\kappa}(x)S_{n\kappa}(x')=0. \tag{10}$$

It is to be noticed that it is sufficient to prove relations (7)–(9) since equation (10) is an immediate consequence of equation (9). We start with the relation (7). Utilizing equations (3)–(5) we may succesively transform its left-hand side as follows

$$\begin{aligned}&\frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty}S_{n\kappa}(x)S_{n\kappa}(x')\\ =\;&\frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty}\frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\end{aligned}$$

$$
\begin{aligned}
\times\quad & \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\\
\times\quad & \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=1}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=0}^{\infty}\frac{(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]
\end{aligned}
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$$
\begin{aligned}
&+ \sum_{n=-\infty}^{0} \frac{(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \quad \frac{1}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left[\sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \right. \\
&+ \quad \left. \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \right] \\
&= \quad \frac{1}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left[\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \right. \\
&+ \quad \left. \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \right] \\
&= \quad x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') = \delta(x-x'), \qquad (11)
\end{aligned}
$$

where the last equality stems from the closure relation (6). Thus we have proved the relation (7).

Consider now the relation (8). Successive transformations of its left-hand side yield

$$
\begin{aligned}
&\frac{\alpha^{-1}\varepsilon^{-1}}{2} \sum_{n=-\infty}^{\infty} T_{n\kappa}(x) T_{n\kappa}(x') \\
&= \quad \frac{\alpha^{-1}\varepsilon^{-1}}{2} \sum_{n=-\infty}^{\infty} \frac{\varepsilon\alpha(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) \right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x') \right] \\
&= \quad \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) \right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x') \right] \\
&= \quad \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left[\sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \right. \\
&- \quad \sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \quad \left. \sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2 L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{\infty} \frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{\infty} \frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{\infty} \frac{(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{-1} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{-1} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{0} \frac{(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') = \delta(x-x'), \qquad (12)
\end{aligned}
$$

where again the last equality follows from the closure relation (6). Thus we have proved correctness of the relation (8).

Finally, successive transformations of the left-hand side of the relation (9) give

$$
\begin{aligned}
&\frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty} S_{n\kappa}(x)T_{n\kappa}(x') \\
&= \frac{\alpha^{-1}\varepsilon}{2}\sum_{n=-\infty}^{\infty}\frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2} \\
&\times \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right] \\
&= \frac{\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \\
&\times \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right] \\
&= \frac{\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right. \\
&- \sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x') \\
&+ \sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&- \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2 L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right] \\
&= \frac{\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right. \\
&- \sum_{n=-\infty}^{\infty}\frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x') \\
&+ \sum_{n=-\infty}^{\infty}\frac{\mp|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&- \left.\sum_{n=-\infty}^{\infty}\frac{(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right] \\
&= \frac{\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=1}^{\infty}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right. \\
&+ \sum_{n=-\infty}^{-1}\frac{(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&+ \sum_{n=1}^{\infty}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x') \\
&- \left.\sum_{n=-\infty}^{-1}\frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{0} \frac{(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\varepsilon}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\varepsilon}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\varepsilon}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] = 0, \qquad (13)
\end{aligned}
$$

where the last equality is obvious. This completes the proof of correctness of the closure relation (1).

Consider now the closure relation (2). It is equivalent to four relations

$$\frac{Z}{x} \sum_{n=-\infty}^{\infty} \mu_{n\kappa} S_{n\kappa}(x) S_{n\kappa}(x') = \delta(x-x'), \qquad (14)$$

$$-\frac{Z}{x} \sum_{n=-\infty}^{\infty} \mu_{n\kappa}^{-1} T_{n\kappa}(x) T_{n\kappa}(x') = \delta(x-x'), \qquad (15)$$

$$\frac{Z}{x} \sum_{n=-\infty}^{\infty} S_{n\kappa}(x) T_{n\kappa}(x') = 0 \qquad (16)$$

and

$$-\frac{Z}{x} \sum_{n=-\infty}^{\infty} T_{n\kappa}(x) S_{n\kappa}(x') = 0. \qquad (17)$$

We observe that the 'off-diagonal' relations (16) and (17) follow immediately from the relations (9) and (10) and therefore it is sufficient to prove the 'diagonal' relations (14) and (15), a task that is a little bit more difficult than proofs of the relations (7) and (8). Upon transforming the left-hand side of equation (14) we have

$$
\begin{aligned}
&\frac{Z}{x} \sum_{n=-\infty}^{\infty} \mu_{n\kappa} S_{n\kappa}(x) S_{n\kappa}(x') \\
&= \frac{Z}{x} \sum_{n=-\infty}^{\infty} \frac{\varepsilon}{\alpha Z} (|n| + \gamma_\kappa \pm N_{n\kappa}) \frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) + \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) \right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') + \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x') \right] \\
&= \frac{1}{2} x^{\gamma_\kappa - 1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{(|n| + \gamma_\kappa \pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)}
\end{aligned}
$$

$$
\begin{aligned}
\times\quad & \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{2}x^{\gamma_\kappa-1}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{2}x^{\gamma_\kappa-1}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa\pm N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa\pm N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa\pm N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{1}{2}x^{\gamma_\kappa-1}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa-N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa-N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa-N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=0}^{\infty}\frac{(|n|+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\kappa)(n-1)!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\kappa)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\kappa+1)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\kappa)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(2n+2\gamma_\kappa+1)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \Bigg] . \qquad (18)
\end{aligned}
$$

Before we shall identify the obtained expression, we shall transform to the same form the left-hand side of the relation (15)

$$
\begin{aligned}
&-\frac{Z}{x} \sum_{n=-\infty}^{\infty} \mu_{n\kappa}^{-1} T_{n\kappa}(x) T_{n\kappa}(x') \\
&= -\frac{Z}{x} \sum_{n=-\infty}^{\infty} \frac{-\varepsilon^{-1}}{\alpha Z} (|n| + \gamma_\kappa \mp N_{n\kappa}) \frac{\alpha\varepsilon(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) \right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') - \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x') \right] \\
&= \frac{1}{2} x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{(|n| + \gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)} \\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) \right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') - \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x') \right] \\
&= \frac{1}{2} x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n| + \gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \left(\frac{\kappa \mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2 L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa - 1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa \mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa \mp N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa \mp N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa \mp N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa - 1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa - N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa + N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa - N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa + N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa - N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa + N_{n\kappa})|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(|n|+\gamma_\kappa - N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa + N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa - 1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa - N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa + N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa - N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{1}{2} x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\kappa)(n-1)!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\kappa)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\kappa+1)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\kappa)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(2n+2\gamma_\kappa+1)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \Bigg] . \qquad (19)
\end{aligned}
$$

Consider now the closure relation (6) in the case $\alpha = 2\gamma_\kappa - 1$. Upon utilizing the recurrence relation

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x) \qquad (20)$$

we may transform it as follows

$$
\begin{aligned}
\delta(x-x') &= x^{\gamma_\kappa-1/2} x'^{\gamma_\kappa-1/2} \mathrm{e}^{-(x+x')/2} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa-1)}(x') \\
&= x^{\gamma_\kappa-1/2} x'^{\gamma_\kappa-1/2} \mathrm{e}^{-(x+x')/2} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa)}(x) - L_{n-1}^{(2\gamma_\kappa)}(x) \right] \left[L_n^{(2\gamma_\kappa)}(x') - L_{n-1}^{(2\gamma_\kappa)}(x') \right] \\
&= x^{\gamma_\kappa-1/2} x'^{\gamma_\kappa-1/2} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') - \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1/2} x'^{\gamma_\kappa-1/2} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') - \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+1)!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= x^{\gamma_\kappa-1/2} x'^{\gamma_\kappa-1/2} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(2n+2\gamma_\kappa+1)n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$-\sum_{n=1}^{\infty}\frac{n!}{\Gamma(n+2\gamma_\kappa)}L_n^{(2\gamma_\kappa)}(x)L_{n-1}^{(2\gamma_\kappa)}(x')-\sum_{n=1}^{\infty}\frac{n!}{\Gamma(n+2\gamma_\kappa)}L_{n-1}^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa)}(x')\Bigg]. \tag{21}$$

Comparison of equations (18), (19) and (21) yields

$$\frac{Z}{x}\sum_{n=-\infty}^{\infty}\mu_{n\kappa}S_{n\kappa}(x)S_{n\kappa}(x')=\sqrt{x'/x}\,\delta(x-x')=\delta(x-x') \tag{22}$$

and

$$-\frac{Z}{x}\sum_{n=-\infty}^{\infty}\mu_{n\kappa}^{-1}T_{n\kappa}(x)T_{n\kappa}(x')=\sqrt{x'/x}\,\delta(x-x')=\delta(x-x') \tag{23}$$

which completes the proofs of the relations (14) and (15) and consequently of the closure relation (2).

2 A transformed form of the radial Dirac-Coulomb Green function

In the Sturmian expansions of the radial Dirac-Coulomb Green function (equations (134) and (135) of Ref. [1]), the elements $g_\kappa^{(ij)}(r,r';E)$ have been expressed as infinite series with summations running over all, positive and negative, integers. By reducing terms with the same $|n|$ to a common denominator, the expansions may be transformed to forms containing only summations over non-negative integers. Definning

$$\epsilon=\frac{E}{mc^2},\qquad \omega=\frac{\zeta\epsilon}{\sqrt{1-\epsilon^2}}, \tag{24}$$

one has

$$\begin{aligned}
g_\kappa^{(11)}(r,r';E) &= \sum_{n=-\infty}^{\infty}\frac{\mu_{n\kappa}}{\mu_{n\kappa}-1}S_{n\kappa}(x)S_{n\kappa}(x')\\
&= \sum_{n=-\infty}^{\infty}\frac{1}{1+\frac{|n|+\gamma_\kappa\mp N_{n\kappa}}{\varepsilon\zeta}}\,\frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\,\frac{|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa}}{(|n|+\gamma_\kappa+\varepsilon\zeta)^2-N_{n\kappa}^2}\\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\,\frac{|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa}}{2\varepsilon\zeta(|n|+\gamma_\kappa-\omega)}\\
&\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= \frac{\alpha}{4\varepsilon}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}
\end{aligned}$$

$$
\begin{aligned}
\times\quad & \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{\alpha}{4\varepsilon}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{\alpha}{4\varepsilon}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
\\
=\quad & \frac{\alpha}{4\varepsilon}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\right.\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
-\quad & \sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
+\quad & \left.\sum_{n=0}^{\infty}\frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{4\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg], \\
&= \frac{\alpha}{4\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa)(n-1)!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa)}(x) - L_n^{(2\gamma_\kappa-1)}(x) \right] L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) \left[L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=0}^{\infty} \frac{(-n-3\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega+1+\varepsilon\zeta+\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&+ \sum_{n=0}^{\infty} \frac{(-n-\gamma_\kappa+\omega+\varepsilon\zeta-\omega-2\gamma_\kappa-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(\varepsilon\zeta+\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(\varepsilon\zeta-\omega-2\gamma_\kappa-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ (\varepsilon\zeta+\omega+\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&+ (\varepsilon\zeta-\omega-2\gamma_\kappa-\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha}{2\varepsilon} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \left(\frac{\zeta}{\sqrt{1-\epsilon^2}} + \kappa \right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&+ \left(\frac{\zeta(1-2\epsilon)}{\sqrt{1-\epsilon^2}} - 2\gamma_\kappa - \kappa \right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \qquad (25)
\end{aligned}
$$

$$
g_\kappa^{(12)}(r,r';E) = \sum_{n=-\infty}^{\infty} \frac{1}{\mu_{n\kappa}-1} S_{n\kappa}(x) T_{n\kappa}(x')
$$

$$
= \sum_{n=-\infty}^{\infty} \frac{1}{\varepsilon \frac{|n|+\gamma_\kappa \pm N_{n\kappa}}{\zeta} - 1} \frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa} \mp \kappa)\Gamma(|n|+2\gamma_\kappa)} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2}
$$

$$
\begin{aligned}
&\times \quad \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa}}{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta)^2-N_{n\kappa}^2}\\
&\times \quad \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2\varepsilon N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa}}{(-2)\varepsilon^{-1}\zeta(|n|+\gamma_\kappa-\omega)}\\
&\times \quad \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad -\frac{\alpha}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\\
&\times \quad \left[L^{(2\gamma_\kappa)}_{|n|-1}(x)+\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)\right]\left[L^{(2\gamma_\kappa)}_{|n|-1}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad -\frac{\alpha}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right.\\
&- \quad \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&+ \quad \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&- \quad \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad -\frac{\alpha}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right.\\
&- \quad \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&+ \quad \sum_{n=-\infty}^{\infty}\frac{\mp(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&- \quad \left.\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right]\\
&= \quad -\frac{\alpha}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\left[\sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\right.\\
&+ \quad \sum_{n=-\infty}^{-1}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&+ \quad \left.\sum_{n=1}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa)(n-1)!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa)}(x) - L_n^{(2\gamma_\kappa-1)}(x) \right] L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) \left[L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega+1-\varepsilon^{-1}\zeta+\omega-\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega-\varepsilon^{-1}\zeta+\omega+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= -\frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(\varepsilon^{-1}\zeta-\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(\varepsilon^{-1}\zeta-\omega-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ (\varepsilon^{-1}\zeta-\omega+\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&- (\varepsilon^{-1}\zeta-\omega-\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \left(\frac{\zeta}{\sqrt{1-\epsilon^2}} + \kappa \right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right]
\end{aligned}
$$

$$- \quad \left(\frac{\zeta}{\sqrt{1-\epsilon^2}} - \kappa\right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}, \tag{26}$$

$$g_\kappa^{(21)}(r,r';E) = \sum_{n=-\infty}^{\infty} \frac{\mu_{n\kappa}}{\mu_{n\kappa}-1} T_{n\kappa}(x) S_{n\kappa}(x')$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{1+\frac{|n|+\gamma_\kappa \mp N_{n\kappa}}{\varepsilon\zeta}} \frac{\alpha(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2}$$
$$\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x)\right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') + \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x')\right]$$

$$= x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{\alpha\varepsilon\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa}}{(|n|+\gamma_\kappa+\varepsilon\zeta)^2-N_{n\kappa}^2}$$
$$\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x)\right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') + \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x')\right]$$

$$= x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{\alpha\varepsilon\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)} \frac{|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa}}{2\varepsilon\zeta(|n|+\gamma_\kappa-\omega)}$$
$$\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x)\right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') + \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x')\right]$$

$$= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}$$
$$\times \left[L_{|n|-1}^{(2\gamma_\kappa)}(x) - \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x)\right] \left[L_{|n|-1}^{(2\gamma_\kappa)}(x') + \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x')\right]$$

$$= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x')$$
$$+ \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x')$$
$$- \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} \frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x')$$
$$- \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} \left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2 L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x') \Bigg]$$

$$= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x')$$
$$+ \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|-1}^{(2\gamma_\kappa)}(x) L_{|n|}^{(2\gamma_\kappa)}(x')$$
$$- \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L_{|n|}^{(2\gamma_\kappa)}(x) L_{|n|-1}^{(2\gamma_\kappa)}(x')$$

$$
\begin{aligned}
&- \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta\pm N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L^{(2\gamma_\kappa)}_{|n|}(x) L^{(2\gamma_\kappa)}_{|n|}(x') \Bigg] \\
&= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|-1}(x) L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|-1}(x) L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&- \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|-1}(x) L^{(2\gamma_\kappa)}_{|n|}(x') \\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|-1}(x) L^{(2\gamma_\kappa)}_{|n|}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|}(x) L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{|n|}(x) L^{(2\gamma_\kappa)}_{|n|-1}(x') \\
&- \sum_{n=0}^{\infty} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L^{(2\gamma_\kappa)}_{|n|}(x) L^{(2\gamma_\kappa)}_{|n|}(x') \\
&- \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)} L^{(2\gamma_\kappa)}_{|n|}(x) L^{(2\gamma_\kappa)}_{|n|}(x') \Bigg] \\
&= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n-1}(x) L^{(2\gamma_\kappa)}_{n-1}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n-1}(x) L^{(2\gamma_\kappa)}_{n-1}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n-1}(x) L^{(2\gamma_\kappa)}_{n}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n-1}(x) L^{(2\gamma_\kappa)}_{n}(x') \\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n}(x) L^{(2\gamma_\kappa)}_{n-1}(x') \\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n}(x) L^{(2\gamma_\kappa)}_{n-1}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L^{(2\gamma_\kappa)}_{n}(x) L^{(2\gamma_\kappa)}_{n}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L^{(2\gamma_\kappa)}_{n}(x) L^{(2\gamma_\kappa)}_{n}(x') \Bigg] \\
&= \frac{\alpha}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L^{(2\gamma_\kappa)}_{n-1}(x) L^{(2\gamma_\kappa)}_{n-1}(x')
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa)(n-1)!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_{n}^{(2\gamma_\kappa)}(x) - L_{n}^{(2\gamma_\kappa-1)}(x) \right] L_{n}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n}^{(2\gamma_\kappa)}(x) \left[L_{n}^{(2\gamma_\kappa)}(x') - L_{n}^{(2\gamma_\kappa-1)}(x') \right] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_{n}^{(2\gamma_\kappa)}(x) L_{n}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

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\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta+\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa+\varepsilon\zeta-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega+1+\varepsilon\zeta+\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&- \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega+\varepsilon\zeta+\omega-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(\varepsilon\zeta+\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big] \\
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(\varepsilon\zeta+\omega-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\} \\
&= \frac{\alpha}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ (\varepsilon\zeta+\omega+\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \Big[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \quad (\varepsilon\zeta+\omega-\kappa)\sum_{n=0}^{\infty}\frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)}L_n^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa)}(x')\Bigg\}\\
&= \quad \frac{\alpha}{2}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\Bigg\{\left(\frac{\zeta}{\sqrt{1-\epsilon^2}}+\kappa\right)\sum_{n=0}^{\infty}\frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)}L_n^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa)}(x')\\
&+ \quad \sum_{n=0}^{\infty}\frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}\left[L_n^{(2\gamma_\kappa-1)}(x)L_n^{(2\gamma_\kappa)}(x')-L_n^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa-1)}(x')\right]\\
&- \quad \left(\frac{\zeta}{\sqrt{1-\epsilon^2}}-\kappa\right)\sum_{n=0}^{\infty}\frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)}L_n^{(2\gamma_\kappa)}(x)L_n^{(2\gamma_\kappa)}(x')\Bigg\} \qquad (27)
\end{aligned}
$$

$$
g_\kappa^{(22)}(r,r';E)=\sum_{n=-\infty}^{\infty}\frac{1}{\mu_{n\kappa}-1}T_{n\kappa}(x)T_{n\kappa}(x')
$$

$$
\begin{aligned}
&= \quad \sum_{n=-\infty}^{\infty}\frac{1}{\varepsilon\frac{|n|+\gamma_\kappa\pm N_{n\kappa}}{\zeta}-1}\frac{\alpha\varepsilon(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= \quad x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa}}{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta)^2-N_{n\kappa}^2}\\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= \quad x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{\alpha\zeta(|n|+2\gamma_\kappa)|n|!}{2N_{n\kappa}(N_{n\kappa}\mp\kappa)\Gamma(|n|+2\gamma_\kappa)}\frac{|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa}}{(-2)\varepsilon^{-1}\zeta(|n|+\gamma_\kappa-\omega)}\\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= \quad -\frac{\alpha\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\\
&\times \quad \left[L_{|n|-1}^{(2\gamma_\kappa)}(x)-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)\right]\left[L_{|n|-1}^{(2\gamma_\kappa)}(x')-\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x')\right]\\
&= \quad -\frac{\alpha\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\Bigg[\sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')\\
&- \quad \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|-1}^{(2\gamma_\kappa)}(x)L_{|n|}^{(2\gamma_\kappa)}(x')\\
&- \quad \sum_{n=-\infty}^{\infty}\frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}L_{|n|}^{(2\gamma_\kappa)}(x)L_{|n|-1}^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}\left(\frac{\kappa\mp N_{n\kappa}}{|n|+2\gamma_\kappa}\right)^2 L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\Bigg]\\
&= -\frac{\alpha\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\Bigg[\sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}\mp\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&- \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&- \sum_{n=-\infty}^{\infty} \frac{\mp(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&+ \sum_{n=-\infty}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta\mp N_{n\kappa})(N_{n\kappa}\mp\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\Bigg]\\
&= -\frac{\alpha\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\Bigg[\sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}-\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&+ \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(|n|+2\gamma_\kappa)|n|!}{N_{n\kappa}(N_{n\kappa}+\kappa)(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&+ \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|-1}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&+ \sum_{n=1}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&- \sum_{n=-\infty}^{-1} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|-1}(x')\\
&+ \sum_{n=0}^{\infty} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\\
&+ \sum_{n=-\infty}^{0} \frac{(|n|+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)|n|!}{N_{n\kappa}(|n|+\gamma_\kappa-\omega)\Gamma(|n|+2\gamma_\kappa+1)}L^{(2\gamma_\kappa)}_{|n|}(x)L^{(2\gamma_\kappa)}_{|n|}(x')\Bigg]\\
&= -\frac{\alpha\varepsilon}{4}x^{\gamma_\kappa}x'^{\gamma_\kappa}\mathrm{e}^{-(x+x')/2}\Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}-\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{n-1}(x)L^{(2\gamma_\kappa)}_{n-1}(x')\\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(n+2\gamma_\kappa)n!}{N_{n\kappa}(N_{n\kappa}+\kappa)(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{n-1}(x)L^{(2\gamma_\kappa)}_{n-1}(x')\\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{n-1}(x)L^{(2\gamma_\kappa)}_{n}(x')\\
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{n-1}(x)L^{(2\gamma_\kappa)}_{n}(x')\\
&+ \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)}L^{(2\gamma_\kappa)}_{n}(x)L^{(2\gamma_\kappa)}_{n-1}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha\varepsilon}{4} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=1}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}+\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}-\kappa)(n-1)!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- 2\sum_{n=1}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-N_{n\kappa})(N_{n\kappa}-\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+N_{n\kappa})(N_{n\kappa}+\kappa)n!}{N_{n\kappa}(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa)(n-1)!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_{n-1}^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_{n-1}^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg] \\
&= -\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa)}(x) - L_n^{(2\gamma_\kappa-1)}(x)\right] L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) \left[L_n^{(2\gamma_\kappa)}(x') - L_n^{(2\gamma_\kappa-1)}(x')\right] \\
&+ \sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left[\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right. \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \\
&+ \left.\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right]
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left\{\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\varepsilon^{-1}\zeta-\kappa+1)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right. \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x')\right] \\
&+ \left.\sum_{n=0}^{\infty} \frac{(-n-3\gamma_\kappa-\varepsilon^{-1}\zeta+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right\}
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left\{\sum_{n=0}^{\infty} \frac{(n+\gamma_\kappa-\omega+1-\varepsilon^{-1}\zeta+\omega-\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right. \\
&+ \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x')\right] \\
&+ \left.\sum_{n=0}^{\infty} \frac{(-n-\gamma_\kappa+\omega-\varepsilon^{-1}\zeta-\omega-2\gamma_\kappa+\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right\}
\end{aligned}
$$

$$
\begin{aligned}
= &-\frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \left\{\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x')\right. \\
&- \sum_{n=0}^{\infty} \frac{(\varepsilon^{-1}\zeta-\omega+\kappa)n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&+ \left.\sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x')\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{(\varepsilon^{-1}\zeta+\omega+2\gamma_\kappa-\kappa)n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ (\varepsilon^{-1}\zeta-\omega+\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&+ (\varepsilon^{-1}\zeta+\omega+2\gamma_\kappa-\kappa) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{\alpha\varepsilon}{2} x^{\gamma_\kappa} x'^{\gamma_\kappa} \mathrm{e}^{-(x+x')/2} \Bigg\{ \left(\frac{\zeta}{\sqrt{1-\epsilon^2}} + \kappa \right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega+1)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \\
&- \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa)} \left[L_n^{(2\gamma_\kappa-1)}(x) L_n^{(2\gamma_\kappa)}(x') + L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa-1)}(x') \right] \\
&+ \left(\frac{\zeta(1+2\epsilon)}{\sqrt{1-\epsilon^2}} + 2\gamma_\kappa - \kappa \right) \sum_{n=0}^{\infty} \frac{n!}{(n+\gamma_\kappa-\omega)\Gamma(n+2\gamma_\kappa+1)} L_n^{(2\gamma_\kappa)}(x) L_n^{(2\gamma_\kappa)}(x') \Bigg\}. \qquad (28)
\end{aligned}
$$

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